Computational topology
Homework 3 (due: January 21st 2025)

Each problem is worth a certain amount of points. The point assignment should somewhat reflect the difficulty of the problem. Some problems are theoretical, others require you also submit the code (that conforms to the requirements given in the problem description). You may choose which problems to solve, **15 points is equal to 100%**.

You have to submit your solutions before the deadline as **one .zip file** to the appropriate mailbox at https://ucilnica.fri.uni-lj.si/course/view.php?id=111 (near the top of the page).

This .zip file should contain:

1. **a namesurname.pdf file written in LaTeX** and containing the solutions to the theoretical problems you have chosen as well as solutions and explanations for the programming problems (also make sure you sign your name on the top of the first page),
2. **.jl files** containing the code (one for each of the programming problems you have chosen).

1 Theoretical problems

1. **(3 points) Homology.**

   A simplicial complex X contains the following maximal simplices:

   $$ADF, BEF, CD, CE, DE.$$

   (a) Draw X as a planar 2-dimensional simplicial complex.

   (b) Write down the chain groups C_n.

   (c) Determine the boundary homomorphisms $\partial_n \colon C_n \to C_{n-1}$.

   (d) Find the cycles Z_n and boundaries B_n.

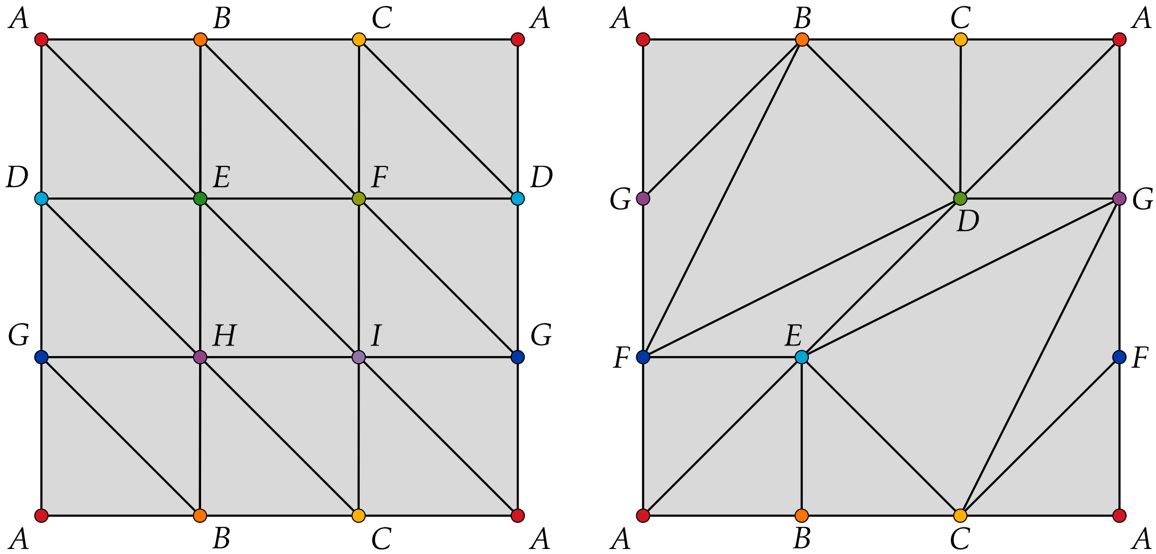
   (e) Determine $H_n(X;\mathbb{Z})$.

   (f) Determine $H_n(X;\mathbb{Z}_2)$.

   (g) Determine the Betti numbers of X and compute the Euler characteristic of X.

2. **(3 points) Discrete Morse Theory.**

   Recall that a torus T is obtained by gluing together the two pairs of opposite sides of the square. Two possible triangulations of the torus are given below. Pick one to use for the rest of this problem.



(a) Construct a Morse function F on T, draw the corresponding vector field and then cancel all possible pairs of critical simplices to minimize the number of critical simplices. (Or try drawing an optimal gradient vector field without constructing the function first.)

(b) Determine the number c_i of critical simplices of dimension i and compute the Euler characteristic $\chi(T)$.

(c) What are the Betti numbers of T with $\mathbb{Z}$ coefficients?

2 Programming problems

3. **(5 points) Čech complex**

   The file `cech.jl` should contain a function `cech(S, epsilon)` that returns a dictionary where keys are the dimensions of simplices in the Čech complex $\check{C}_\epsilon(S)$ and values are lists of all simplices of corresponding dimension.

   Sample input for `cech(S, epsilon)`:

```
S = [(-2, 1), (-2, -2), (1, -1), (1.5, 2.5)]
epsilon = 2
```

   Sample output for `cech(S, epsilon)`:

```
0 => [(1,), (2,), (3,), (4,)],
1 => [(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)],
2 => [(1, 2, 3)]
```

   You can implement this algorithm by first building the proximity graph and then successively add simplices of higher dimensions. You are free to use Julia package BoundingSphere, but you should implement other parts of the algorithm yourself. (You can assume that the line '`using BoundingSphere`' will be present in my Julia test environment.)

4. **(4 points) Collapsibility**

   Write an algorithm that takes a simplicial complex given as a list of maximal simplices (which are not necessarily all of the same dimension) and simplifies it by collapsing any free faces.

Your file `collapse.jl` should contain a function `collapse(X, progress = false)` that returns the list of all simplices that are left after all possible collapses have been made. If the optional parameter is `true`, it prints the progress report to the console. Here is the first few lines of output for the cylinder:

```
Initial simplicial complex:
[(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (1, 5, 6), (1, 2, 6)]

Free faces:
[((1, 2, 3), (1, 3)), ((4, 5, 6), (4, 6)), ((1, 5, 6), (1, 5)),...]
Choose a simplex sigma with a free face tau:
sigma = (1, 2, 3)
tau = (1, 3)
Remaining simplices after the elementary collapse:
[(2, 3, 4), (3, 4, 5), (4, 5, 6), (1, 5, 6), (1, 2, 6)]
...
```

Run it for a 2-sphere, a cylinder, a Moebius strip and the Dunce hat given below. Did you get the expected results?

```
S2 = [(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)]
C = [(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (1, 5, 6), (1, 2, 6)]
M = [(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (2, 5, 6), (1, 2, 6)]
D = [(1, 2, 5), (1, 5, 6), (1, 6, 7), (1, 2, 7), (1, 4, 9),
     (1, 9, 10), (1, 10, 11), (1, 4, 11), (1, 2, 13), (1, 13, 14),
     (1, 14, 15), (1, 4, 15), (2, 3, 5), (2, 3, 7), (3, 4, 9),
     (3, 4, 11), (2, 3, 13), (3, 4, 15), (3, 7, 8), (3, 8, 9),
     (3, 11, 12), (3, 12, 13), (3, 15, 16), (3, 5, 16),
     (5, 6, 17), (5, 16, 17), (6, 7, 17), (7, 8, 17),
     (8, 9, 17), (9, 10, 17), (10, 11, 17), (11, 12, 17),
     (12, 13, 17), (13, 14, 17), (14, 15, 17), (15, 16, 17)]
```

Finally, try an example where maximal simplices have different dimensions.

```
X = [(1, 2, 3), (2, 3, 5), (3, 4), (5, 6)]
```

Include the sequence of collapses for X in your report and come up with a few more test cases.