

Probabilistic Analysis and Randomized Algorithms



Prof Marko Robnik-Šikonja

Analysis of Algorithms and Heuristic Problem Solving
Edition 2025

Finding maximum

```
findMax(n) {  
    fbest =  $-\infty$  ;  
    for (i=1 ; i <= n ; i++) {  
        fi = check(A[i]) ;  
        if (fi > fbest) {  
            fbest = fi ;  
            process(A[i]) ;  
        }  
    }  
}
```

- $O(n \cdot c_{\text{check}} + m \cdot c_{\text{process}})$
- worst case analysis
- probabilistic analysis
- randomization

Probabilistic analysis

- assumptions about the input distributions
- indicator random variables

Randomization

- to avoid “bad” input sequences, we intentionally randomize the input

```
void findMax(n) {  
    randomly shuffle elements in A  
    fbest = 0 ;  
    for (i=1 ; i <= n ; i++) {  
        fi = check(A[i]) ;  
        if (fi > fbest) {  
            fbest = fi ;  
            process(A[i]) ;  
        }  
    }  
}
```

Randomize the input

PERMUTE-BY-SORTING(A)

```
1   $n = A.length$   
2  let  $P[1..n]$  be a new array  
3  for  $i = 1$  to  $n$   
4       $P[i] = \text{RANDOM}(1, n^3)$   
5  sort  $A$ , using  $P$  as sort keys
```

Randomize the input

RANDOMIZE-IN-PLACE(A)

```
1   $n = A.length$   
2  for  $i = 1$  to  $n$   
3      swap  $A[i]$  with  $A[\text{RANDOM}(i, n)]$ 
```

On-line maximum

- on-line maximum: elements arrive one by one, randomly shuffled; we can check them but we can select only one

Find online maximum

```
findMaxOnline(k, n) {  
    fbest =  $-\infty$  ;  
    for (i=1 ; i <= k ; i++) {  
        if (score(i) > fbest)  
            fbest = fi ;  
    }  
    for (i=k+1 ; i <= n ; i++) {  
        if (score(i) > fbest)  
            return(i) ;  
    }  
    return(n) ;  
}
```

- How to select k, that we shall select the best one with the largest probability?
- What is the probability that we select the best one using this strategy?

Summation bounds

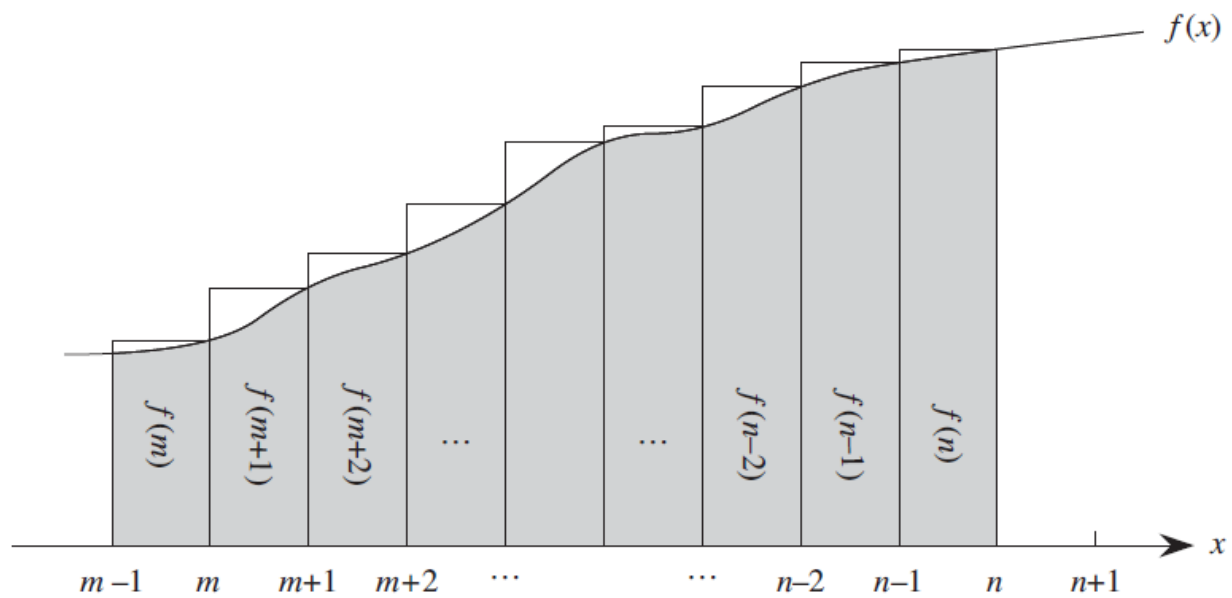
- The summation $\sum_{k=m}^n f(k)$ of monotonously increasing function $f(x)$

on an interval from m to n can be bounded by integrals

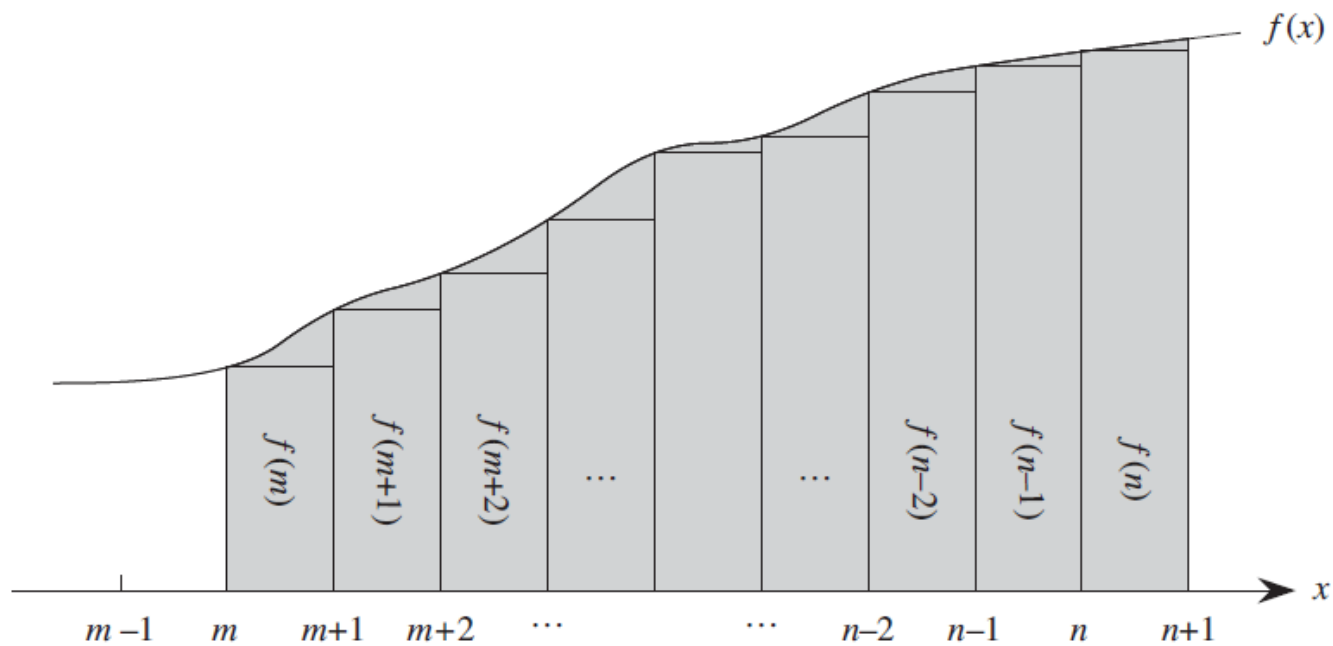
$$\int_{m-1}^n f(x) dx \leq \sum_{k=m}^n f(k) \leq \int_m^{n+1} f(x) dx$$

- The following figures give an explanation

Lower bound



Upper bound



Monotonically decreasing function

- Similarly to monotonically increasing function, we can show the following relation for monotonically decreasing function

$$\int_m^{n+1} f(x) dx \leq \sum_{k=m}^n f(k) \leq \int_{m-1}^n f(x) dx$$

Bounding harmonic series

- In our proof, we used harmonic series which is monotonically decreasing; therefore

$$\int_k^n \frac{1}{x} dx \leq \sum_{i=k}^{n-1} \frac{1}{i} \leq \int_{k-1}^{n-1} \frac{1}{x} dx$$

Graph min-cut

Contraction algorithm:

```
repeat {  
    select random edge  $e=(u,v)$   
    contract  $e$ :  
        replace  $u$  and  $v$  with super-node  $w$   
        keep connections of  $u$  and  $v$  also for  $w$   
        keep parallel edges, but not loops  
}  
until (graph has only two nodes  $v_1$  and  $v_2$ )  
return cut defined by  $v_1$ 
```

- randomized algorithm
- probabilistic analysis

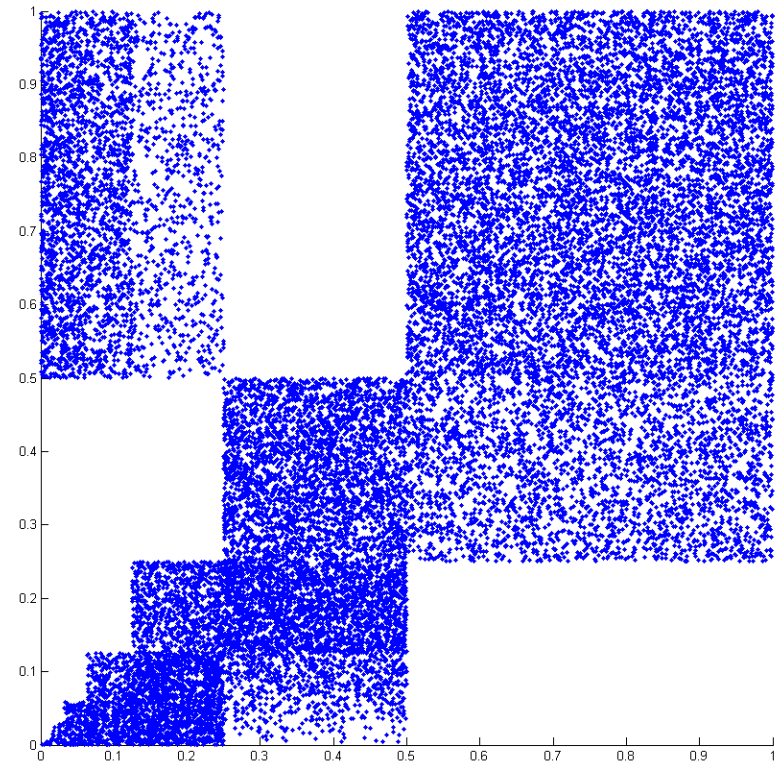
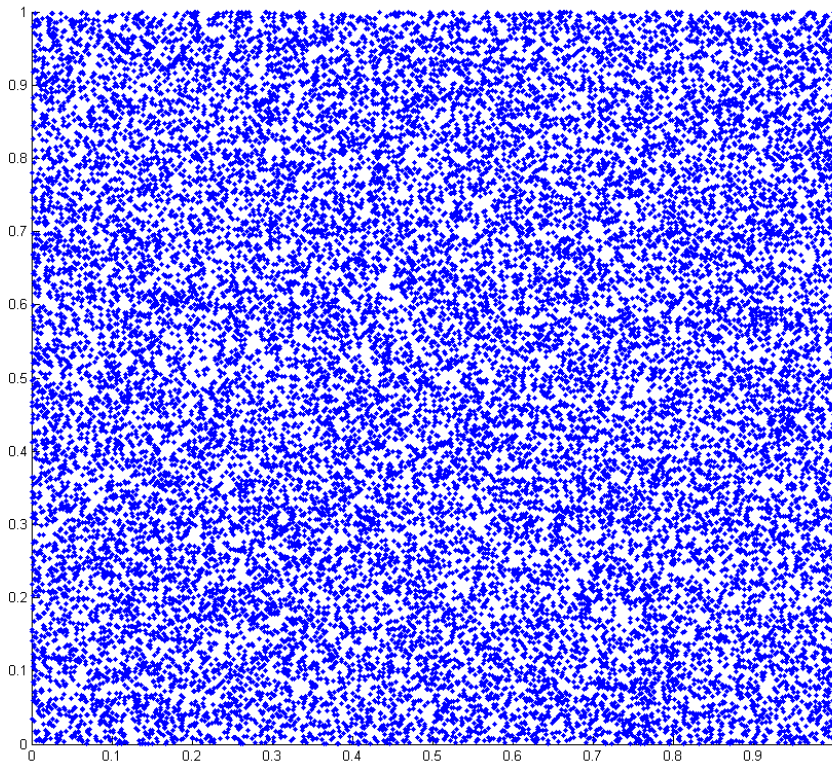
Introduction to pseudo-random numbers

Applications of pseudo random numbers

- computer simulations
- cryptography
- statistical sampling and estimation
- Monte Carlo methods
- data analysis and modelling
- computer games
- games of chance
- hardware and software generators
- quality of (pseudo)random numbers: speed and randomness

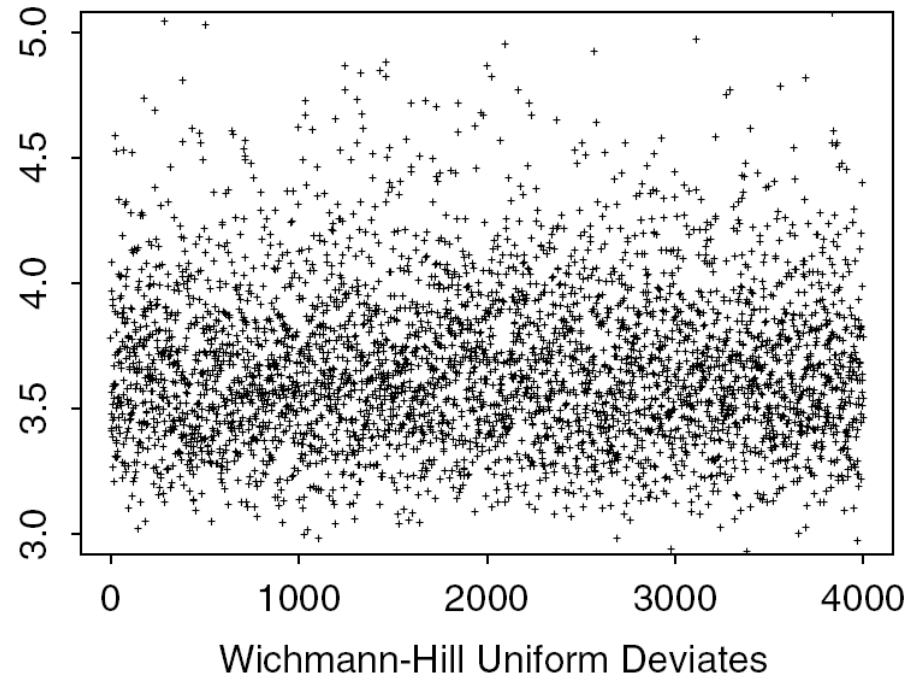
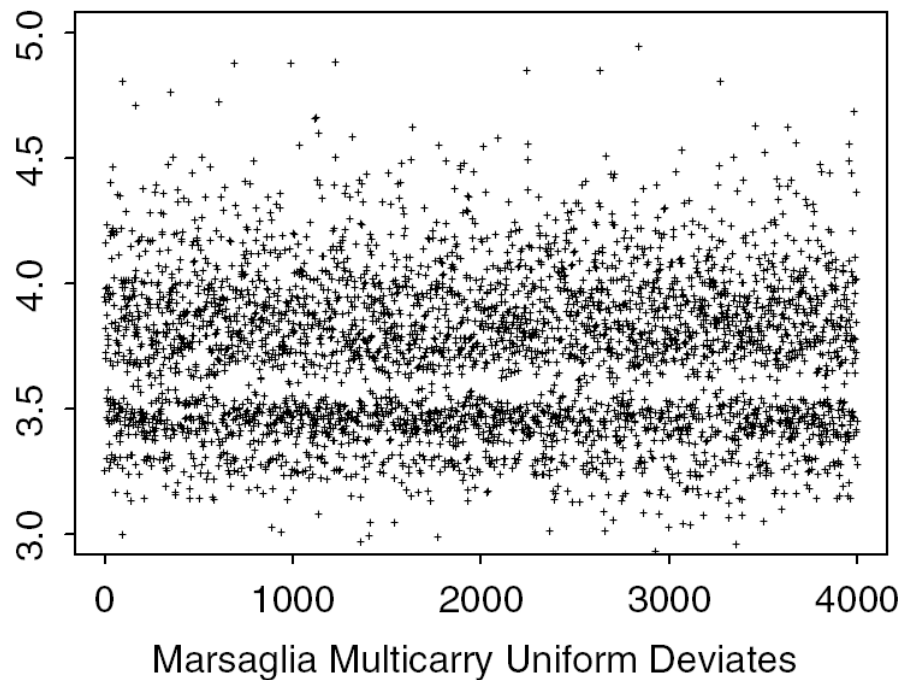
Matlab example

```
Z = rand(28,100000);  
condition = Z(1,:) < 1/4;  
scatter(Z(16,condition),Z(28,condition),'');
```



- P. Savicky: A strong nonrandom pattern in Matlab default random number generator. Technical Report, Institute of Computer Science, Academy of Sciences of Czech Republic (2006)

Example



- Value-at-Risk (financial analysis)
B. D. McCullough: A Review of TESTU01.
Journal of Applied Econometrics, 21: 677–682 (2006)

Quality criteria

- randomness
- speed of generator
- period

Linear congruential generators

- simplest and most common

$$x_i = (a \cdot x_{i-1} + c) \bmod m \quad u_i = x_i / m$$

- A notorious example:

RANDU:

$$x_i = 65539 \cdot x_{i-1} \bmod 2^{31}$$

- simple but bad

MINSTD

- used as a standard for a long time

$$x_i = 16807 \cdot x_{i-1} \bmod (2^{31}-1)$$

i	x_i decimal	x_i binary
1	1	1
2	16807	100000110100111
3	282475249	10000110101100011101011110001
4	1622650073	1100000101101111010110011011001
5	984943658	111010101101010000110000101010
6

Combined linear congruential generator

- combinations of linear congruential generators
- improvements: addition, subtraction, bit mixing
- better randomness, small period

Multiple recursive generators

- higher order recursions

$$x_i = (a_1 \cdot x_{i-1} + \dots + a_k \cdot x_{i-k}) \bmod m$$

$$u_i = x_i / m$$

- e.g., (Knuth, 1998):

$$x_i = (271828183 \cdot x_{i-1} + 314159269 \cdot x_{i-2}) \bmod (2^{31}-1)$$

- combined multiple recursive generators

Other generators

- combinations
- non-linear generators (quadratic, multiplicative, floating point generators, inverse generators)
- (linear) recursive bit generators (modulo 2, operators)
- cryptographic (ISAAC, AES, BBS,...)
- AES [http://en.wikipedia.org/wiki/Advanced Encryption Standard](http://en.wikipedia.org/wiki/Advanced_Encryption_Standard)

BBS (Blum-Blum-Shrub)

- bit generator
- select two large prime integers p and q (e.g., at least 40 decimal places)
- let $m = pq$
- $X_i = X_{i-1}^2 \bmod m$
- $b_i = \text{parity}(X_i)$ (0 if even, 1 if odd)
- finding dependency is equivalent to factorization of m (finding multipliers p and q).
- Currently there is no polynomial non-quantum algorithm for integer factorization but it is not proven that such an algorithm does not exist
- the numbers are therefore currently random enough for most uses

Criteria of randomness

- generate a sequence of t numbers, $u_i \in (0, 1)$
- hypothesis
 u_0, u_1, \dots, u_{t-1} are independent uniformly distributed random variables $U(0,1)$
- equivalent:
vector $(u_0, u_1, \dots, u_{t-1})$
is uniformly randomly distributed in unit hypercube $(0,1)^t$
- equivalent: sequence of independent random bits

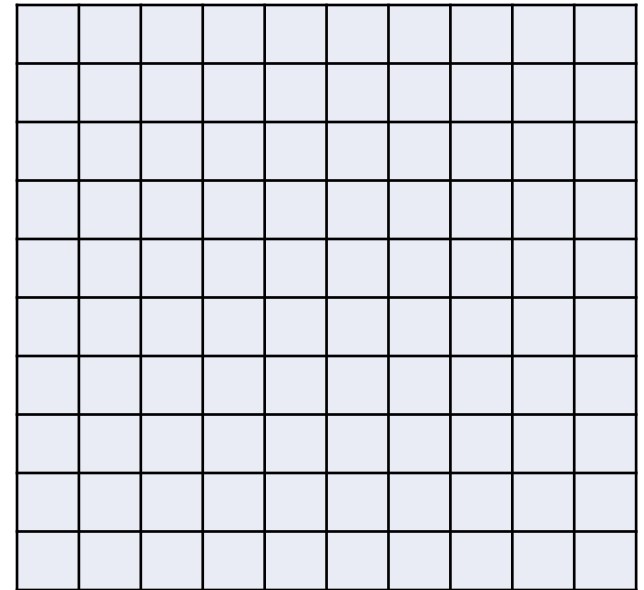
Statistical tests for randomness

- infinitely many possible tests
- only show dependencies, cannot prove that dependencies do not exist
- increase of trust
- *“The difference between the good and bad RNGs, in a nutshell, is that the bad ones fail very simple tests whereas the good ones fail only very complicated tests that are hard to figure out or impractical to run.”*

L'Ecuyer and Simard, 2007. TestU01: A C Library for Empirical Testing of Random Number Generators. *ACM Transactions on Mathematical Software*.

An example of a test

- Pearson's χ^2 goodness-of-fit test
- put generated numbers into k cells (e.g., two-dimensional grid)
- for each cell we know the expected number of elements E_i
- let O_i be the observed number of samples from each cell
- statistics



$$\chi^2_0 = \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i}$$

- if hypothesis of uniform distribution of numbers is true, the statistics χ^2_0 is chi-squared distributed with $k-1$ degrees of freedom
- we reject the hypothesis if $\chi^2_0 > \chi^2_{\alpha, k-p-1}$

Ideas of statistical tests

- one sequence of numbers:
 - tests of groups,
 - gaps,
 - increasing subsequences
- several sequences, hypercube partitioning
 - statistics on partitions
 - statistics on distances
- one sequence of bits
 - cryptographic tests,
 - compressiveness,
 - spectral tests (Fourier),
 - autocorrelation
- several bit sequences

A toolbox of tests

- L'Ecuyer and Simard, 2007. TestU01: A C Library for Empirical Testing of Random Number Generators. *ACM Transactions on Mathematical Software*.
<http://simul.iro.umontreal.ca/testu01/tu01.html>
- results: not many generators pass all tests
- poor results for some popular software (Excel, MATLAB, Mathematica, Java)
- improvements in recent years, e.g.,
<https://www.pcg-random.org/>
- hardware generators